

First Quarterly Report
for
The Development of Computational Techniques for the Identification
of Linear and Nonlinear Mechanical Systems
Subject to Random Excitation

Contract No. NAS5-10106

GPO PRICE \$ _____

CFSTI PRICE(S) \$ _____

Hard copy (HC) 2.00

Microfiche (MF) .50

7 653 July 65

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March 1966

for
Goddard Space Flight Center
Greenbelt, Maryland

N66 26702

FACILITY FORM 602

(ACCESSION NUMBER)
41
(PAGES)
CP 75768
(NASA CR OR TMX OR AD NUMBER)

(THRU)
1
(CODE)
32
(CATEGORY)

ON THE IDENTIFICATION OF LINEAR AND NONLINEAR STRUCTURAL SYSTEMS

Objective: The development of computational techniques for the identification of linear and nonlinear mechanical systems subject to random excitation.

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Summary: A computational procedure has been suggested to determine the differential equation governing the motion of linear and nonlinear structural systems subject to random excitation when the system excitation and response are observed. Such a procedure can yield the transfer functions, impedances and damping coefficients of linear systems as well as determine the nonlinearities in the spring and damping coefficients governing the motion of nonlinear structures.

This report includes a description of the procedural approach taken to the identification of structural systems as well as a detailed description of the quasilinearization-least squares-stagewise smoothing parameter estimation procedure that occupies a central role in the computational procedure. Preliminary computational results illustrating the identification of a simulated one degree of freedom system achieved by the methods suggested are also presented. The anticipated activity in the next quarterly interval will be to verify the computational procedure for multi-degree of freedom systems and to examine the performance of computational model hypothesis testing procedures.

Author

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1. A DESCRIPTION OF THE APPROACH.

The objective of the investigation is to develop a computational procedure for the identification of mechanical structures that are driven by a random excitation. In particular, the structures can be conceived of as an arbitrary collection of lumped spring-mass-damper systems, i. e. an n degree of freedom system in which the springs and dampers may be nonlinear. The system is identified by specifying the number of degrees of freedom and the spring-mass-damper coefficients in the linear case, or a polynomial description of the nonlinearities in the nonlinear case.

The approach employed for the identification of the unknown structure consists of 3 stages. The first is the generation of hypotheses concerning the number of degrees of freedom of the system and the form of the nonlinearities. In effect, this prescribes a conceptual and computational model for the system. In the second stage, the observed data, corresponding to the excitation and response of the system, is used to determine parameters or coefficients of the model assumed to represent the system. The final stage consists of a verification of the validity of the assumed computational model. This is to be accomplished by comparing the response of the system model to the response of the actual system. Subject to an "energy" response criterion, the assumed model is either accepted or an alternative model is assumed and computed on. In case of the latter alternative; the procedure is iterated, starting once again with stage 1.

The principle effort in the investigation is in the development of suitable computational procedures to accomplish the parameter estimation, i. e. the fit of the model to the observed data, specified as stage 2. The technique to be explored for this purpose involves the incorporation of least squares and sequential estimation procedures into the quasilinearization method of system identification.

Very briefly, the quasilinearization procedure is an extension of the Newton-Raphson method of finding the roots of an equation to the problem of finding a piecewise linear (and hence, linear time varying) computational equivalent of a system of nonlinear differential equations. In our problem, the nonlinear system of differential equations is the model assumed to represent the structure. The parameters or coefficients of the model are, in fact, unknown and an initial guess is made of these parameters to permit computation of the assumed model response (with the guessed parameters) to the system excitation. In the quasilinearization procedure, a sequence of observations are made of the system displacement, (the minimum number of observations made is equal to the number of unknown system parameters), and these observations are used in conjunction with a corresponding set of observations on the assumed model to improve the guess or estimate of the unknown model parameters. The computational procedure can be iterated; it has the very attractive computational feature of quadratic convergence, and this is in fact derived from the correspondence of the procedure to the Newton-Raphson method. In effect, the quasilinearization procedure accomplishes system identification by solving a multipoint boundary value problem.

One limitation to the application of the quasilinearization procedure to the identification problem is that the minimum number of observations of the system response may not be sufficient to uniquely specify a solution of the system equation. For example, the solution of the second order linear differential equation that characterizes a one degree of freedom system may pass through two particular displacements at two different time instants for an infinite number of one degree of freedom systems. For this reason, as well as the fact that the observations may be noisy, the least squares technique of parameter estimation is employed to permit more than the minimum number of observations to be employed to enhance our estimate of the unknown parameters.

One complication introduced by using the least squares procedure is that the estimation of parameters by this procedure involves inverting a matrix whose size increases as the number of observations increase. To circumvent the increase of computational time and effort required for an increasingly large number of data points, we resort to stagewise estimation procedure which obviates the requirement for matrix inversion.

A more thorough description of the quasilinearization, least squares and successive approximation procedures appear in section (3).

2. DESCRIPTION OF THE EFFORT AND THE RESULTS FOR THE FIRST QUARTERLY INTERVAL.

The principle effort in this quarterly interval has gone into an attempt to implement the digital computer programs to achieve the quasilinearization - least squares and stagewise smoothing required for the proposed system identification procedure. Also required are digital computer programs to simulate the performance of mechanical structures driven by random excitation.

To date, the computational procedure has been successfully applied to the identification of a one degree of freedom system with an assumed one degree of freedom system model. Some typical computational procedures and results are described and illustrated in section (4).

3. ON THE QUASILINEARIZATION - LEAST SQUARES-STAGEWISE SMOOTHING PROCEDURES.

3.1 History.

The quasilinearization technique is reputedly due to Hestenes at the Rand Corporation in 1949¹. The mathematical theory was investigated by Bellman in 1955² and by Kalaba in 1959³.

In the recent book by Bellman and Kalaba⁴, a historical mathematical perspective is presented which identifies the quasilinearization technique with mathematical activities in geometric duality theory, the calculus of variations, differential inequalities and the function space approximations by Kantorovich.

A first attempt at an application of the quasilinearization technique to the identification problem was made by Shridhar and Kumar in 1964⁵ and a discussion of the application of the least squares procedure to quasilinearization was provided by Lavi and Strauss in 1965⁶. A very significant application of the least squares theory and stagewise estimation procedure was made by Swerling⁷ in 1959 and it has since been reinterpreted and refined by Ho⁸ and Lee and Ho⁹ in 1964. Hypothesis testing procedures are classically a part of the subject of statistical inference. Consequently, literature on this subject is adequately treated in numerous texts. For example, see Middleton¹⁰.)

At present, no comprehensive work exists which permits the quasilinearization-least squares-stagewise smoothing and hypothesis testing procedures to be systematically employed for engineering use in the identification of linear and nonlinear systems. The objective of our effort is then to fill that gap and to achieve a practical implementation of the computational procedures that will permit useful identification of linear and nonlinear structural systems.

3.2 Introduction to Quasilinearization.

3.2.1 Background, Newton's Method.

The concept underlying the technique of quasilinearization is essentially that of Newton's method¹¹ of finding the roots of the solution of an equation. For completeness of exposition, the Newton method is outlined below.

Assumed that we are given the equation*

$$f(x) = 0 \tag{1}$$

and we wish to determine the roots of this equation. First guess solution to this equation say, x_0 , and assume x_0 is such that $f(x_0) \neq 0$. That is, x_0 does not satisfy the equation. The Newton technique is an iterative means of obtaining the roots of (1). Rewrite (1) in the form

$$f(x_0 + e) = 0 \tag{2}$$

where e is an "error" term and expand (2) in a Taylor series around the point x_0 . Equation (2) can therefore be written as

$$f(x_0) + ef'(x_0) + \frac{1}{2} e^2 f''(x_0) + \dots = 0 \tag{3}$$

Designate the error term e by $e = x_1 - x_0$ where x_1 is to be the next "guess" to the solution of (1). Then keeping only the linear term,

* In this report, equations are referred to by a system analagous to the Dewey Decimal System. Equations are numbered starting with the number 1 in each section. They are referred to by that number by statements in the same section. References to equations in other sections have the numerical prefix corresponding to the section in which the equation appears.

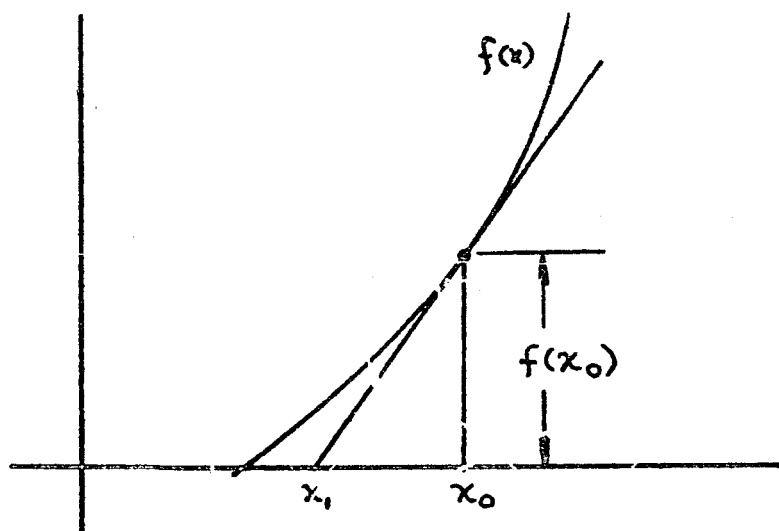
(3) can be rewritten as

$$f(x_0) + (x_1 - x_0) f'(x_0) = 0 \quad (4)$$

Solving (4) for the next guess, x_1 yields

$$x_1 = x_0 - f(x_0) / f'(x_0) \quad (5)$$

Graphically, the situation is depicted in Figure 1.



$$f(x_0) = f'(x_0)(x_0 - x_1)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Figure 1. ILLUSTRATION OF THE NEWTON METHOD

The procedure is iterated, the $n+1$ st approximation to the root of equation (1) is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (6)$$

The procedure converges quadratically, which means that with the root of (1) given by x_r and k a constant (independent of n)

$$x_{n+1} - x_r \leq k |x_n - x_r|^2 \quad (7)$$

A more detailed discussion of the convergence properties of the Newton iteration procedure appears in references 4 and 11.

The quasilinearization technique is a function space generalization of Newton's method.

3.2.2 The One Dimensional Problem.

In this section, the quasilinearization solution of a boundary value problem is demonstrated to provide a computational algorithm for the solution of the identification problem. For illustrative purposes, a first order nonlinear differential equation of known form with an unknown parameter is identified.

Assume that we are given the equation

$$g(x, \dot{x}; t) = 0 \quad (1)$$

which is nonlinear and is known to within a parameter. In this case, assume the unknown parameter is the initial condition, x_0 . Also, assume that a single observation is made of the solution, say for example at $t = 5$, it is known or observed that $x(5) = c$ where c is some particular number. Our objective is to estimate the unknown initial condition x_0 using the quasilinearization technique.

An equivalent form of (1) is

$$\dot{x} = f(x; t) \quad (2)$$

Proceeding in a manner similar to that of the Newton iteration technique, guess a solution, say $x_0(t)$, to (2). Also, assume the solution, $x_0(t)$, is in error by an amount $e(t)$.

The guessed solution is in the form

$$\dot{x}_0(t) + \dot{e}(t) = f(x_0(t) + e(t); t) \quad (3)$$

Then to within second order terms, the Taylor series expansion of (3) is

$$\dot{x}_0(t) + \dot{e}(t) = f(x_0(t); t) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0(t)} e(t) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_0(t)} e^2(t) + \dots \quad (4)$$

Now truncate the expansion after the first term and identify $e(t)$ as the difference between the first guess $x_0(t)$ of the solution to (1) and the second guess $x_1(t)$. That is, let

$$e(t) = x_1(t) - x_0(t) \quad (5)$$

Applying (5) to (4) yields

$$\dot{x}_1 = x_1 \left. \frac{\partial f}{\partial x} \right|_{x=x_0(t)} + f(x_0(t); t) - x_0 \left. \frac{\partial f}{\partial x} \right|_{x=x_0(t)} \quad (6)$$

Equation (6) is rewritten in the form of the time varying linear equation

$$\dot{x}_1 = a_1(t)x_1 + b_1(t) \quad (7)$$

$$\text{where } a_1(t) = \left. \frac{\partial f}{\partial x} \right|_{x=x_0(t)} ; \quad b_1(t) = f(x_0(t); t) - x_0 \left. \frac{\partial f}{\partial x} \right|_{x=x_0(t)}$$

Now, since $x_0(t)$ is known, both $a_1(t)$ and $b_1(t)$ are known and the solution to (7) is

$$x_1(t) = X_0 e^{A_1(t)} + e^{A_1(t)} \int_0^t e^{-A_1(\lambda)} b_1(\lambda) d\lambda \quad (8)$$

where $A_1(t) = \int_0^t a_1(t') dt'$. For later reference, (8) is written in the state space notation₁₂.

$$x_1(t) = \Phi_1(t, 0)x_0 + \int_0^t \Phi_1(t, \lambda) b_1(\lambda) d\lambda \quad (9)$$

In (9), $\Phi_1(t, 0) = e^{A_1(t)}$ and x_0 is the initial condition $x(t=0)$. The quantity $\Phi_1(t, t_0)$ is known as the fundamental or transition matrix.

The fundamental matrix $\Phi(t, t_0)$ is determined from the complementary solution of (7). This solution is in the form

$$x_1(t) = \Phi(t, t_0) x_1(t_0) \quad (10)$$

Equation (10) exhibits the role of the transition matrix $\Phi(t, t_0)$ in transforming the system behavior or state at time t_0 to its state at time t .

Differentiating both sides of (10),

$$\dot{x}_1(t) = \dot{\Phi}(t, t_0) x_1(t_0) \quad (11)$$

and applying (10) and (11) to the complementary differential equation in (7) yields

$$\dot{\Phi}(t, t_0) = a_1(t) \Phi(t, t_0) \quad (12)$$

The quantity $a_1(t)$ is known, therefore, with $\Phi(t, t) = I$, the identity matrix, (12) can be solved for the fundamental matrix $\Phi(t, t_0)$. Then, with $b_1(t)$ known, the right hand integral in (9) can be computed. This we assume done.

Now recall that

(i) we have guessed the solution $x_0(t)$ to the original nonlinear equation (2)

(ii) we have used this guess to determine the linear time varying differential equation approximation to (2) given by (7)

(iii) the solution to (7) the approximate equation is given by (9)

(iv) we have knowledge of a single point observation of the true solution to (2), say at time $t = 5$ i.e. We know $x(5)$.

We require the solution (9) to satisfy the observation $x(5)$. That is, we make $x_1(t)$ identically equal to $x(5)$ and write (9) at time $t = 5$ in the form

$$\Phi_1(5, 0)x_0 = x(5) - \int_0^5 \Phi_1(5, \lambda) b_1(\lambda) d\lambda \quad (13)$$

Equation (13) can be solved for the unknown initial value $x(0) = x_0$, and this value can be used in (9) to generate the solution $x_1(t)$ for all t . This solution $x_1(t)$ now constitutes a new guess of the solution of (2) and the computation process can be repeated to determine the new quantities $a_2(t)$ and $b_2(t)$. Successive solutions $x_3(t)$, $x_4(t)$ etc. can be computed until these solutions differ by an arbitrarily small amount.

3.3 Quasilinearization Applied to the Identification of Nonlinear Mechanical Systems.

In this section we consider the identification of the parameters of a one degree of freedom nonlinear system. The differential equations of motion of the system is given by

$$M \frac{d^2 y}{dt^2} + k_1 \left(1 + a_1 \left(\frac{dy}{dt} \right)^2 \right) \frac{dy}{dt} + k_2 (1 + a_2 y^2) y = x(t) \quad (1)$$

Assume that the driving force $x(t)$ and the mass are known (and the mass is normalized to $M=1$) but that the linear and nonlinear damping and spring coefficients k_1 , a_1 , k_2 , a_2 are unknown.

Equation (1) can be rewritten in state vector form in the following way. Let $y_1 = y$ and $y_2 = \dot{y}$, then

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= -[k_1(1+a_1 y_2^2)y_2 + k_2(1+a_2 y_1^2)y_1] + x(t)\end{aligned}\quad (2)$$

The unknown coefficients are constants, consequently they satisfy the following differential equations.

$$\dot{k}_1 = 0, \quad \dot{a}_1 = 0, \quad \dot{k}_2 = 0, \quad \dot{a}_2 = 0 \quad (3)$$

It is convenient to adjoin the set (3) to the set (2) with the notation

$$q^1 = y_1, \quad q^2 = y_2, \quad q^3 = k_1, \quad q^4 = a_1, \quad q^5 = k_2, \quad q^6 = a_2. \quad (4)$$

A new set of 6 first order nonlinear differential equations in the state variable vector components q^1, \dots, q^6 may be written

$$\dot{q}^1 = q^2,$$

$$\dot{q}^2 = q^3 [1 + q^4 (q^2)^2] q^2 - q^5 [1 + q^6 (q^1)^2] q^1 + x(t), \quad (5)$$

$$\dot{q}^3 = 0, \quad \dot{q}^4 = 0, \quad \dot{q}^5 = 0, \quad \dot{q}^6 = 0$$

where $q(0)$, the initial condition vector is assumed to be unknown.

In vector form (5) may be written

$$\dot{q} = f(q(t); t) + bx(t); q(t=0) = q_0 \quad (6)$$

Compare (6) with equation (3.2.2.7). We will now linearize the vector of functions $f(q(t); t)$ so that (6) will become the vector matrix equivalent of the form (3.2.2.7).

Assume a solution to (6) and designate this, $q_0(t)$. Now consider a function space Taylor series expansion of (6) around the assumed solution $q_0(t)$ in the form.

$$\dot{q}_1 = f(q_0(t); t) + \left. \frac{\partial f}{\partial q} \right|_{q=q_0(t)} [q_1(t) - q_0(t)] + \dots + b x(t). \quad (7)$$

Equation (7) may be rearranged neglecting terms of second and higher order in the form

$$\dot{q}_1(t) = J[f(q_0(t); t)] q_1(t) + \gamma[q_0(t); t] \quad (8)$$

where

$$\gamma[q_0(t); t] = f(q_0(t); t) - J[f(q_0(t); t)] q_0(t) + b x(t) \quad (9)$$

In (9), $J[f(q_0(t); t)]$ is the Jacobian matrix

$$J[f(q_0(t); t)] = \left. \frac{\partial f}{\partial q} \right|_{q=q_0(t)} \quad (10)$$

or

$$J[f(q_0(t); t)] = \begin{bmatrix} \frac{\partial f_1}{\partial q^1} & \frac{\partial f_1}{\partial q^2} & \dots & \frac{\partial f_1}{\partial q^6} \\ \frac{\partial f_2}{\partial q^1} & \frac{\partial f_2}{\partial q^2} & \dots & \frac{\partial f_2}{\partial q^6} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_6}{\partial q^1} & \dots & \dots & \frac{\partial f_6}{\partial q^6} \end{bmatrix} \bigg|_{q = q_0(t)} \quad (11)$$

The Jacobian matrix for the system given in (1) appears on the next page. Equation (8) is now the vector-matrix equivalent of (3.2.2.7) and can be solved in similar manner.

For both systems (8) and (3.2.2.7), observation of a single state variable is assumed to be available. While this is sufficient in the one dimensional situation to generate an estimate of the single component x_0 the initial condition x_0 , that is not the case for (8). In order to generate an estimate of the unknown 6 components initial condition vector required for the solution of (8), it is necessary to make observations of the single available state of (8) at six different times. The details of the solution process, (this corresponds to the solution of a multipoint boundary value problem) will now be examined.

The solution of (8) can be put in the form

$$q_1(t) = q_1^C(t) + c_1^P(t) \quad (12)$$

$$J[f(q_0(t;t))] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -q^5 [1+3q^6(q^1)^2] - q^3 [1+3q^4(q^2)^2] & -[1+q^4(q^2)^2] & q^2 & -q^3(q^2)^3 & -[1+q^6(q^1)^2] & q^1 - q^5(q^1)^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The form of the Jacobian matrix for the particular one degree of freedom system considered

where $q_1^C(t)$, $q_1^P(t)$ respectively designate the complementary and particular solution vectors. Alternatively, we may write the solution to (8) in the state vector form

$$q_1(t) = \Phi_1(t, t_0) q_1(t_0) + \int_{t_0}^t \Phi_1(t, \lambda) \gamma(\lambda) d\lambda \quad (13)$$

where $\Phi_1(t, t_0)$ is the fundamental matrix.

Identify the particular solution, $q_1^P(t)$, as a solution of the differential equation

$$\dot{q}_1^P(t) = J[f(q_0(t); t)] q_1^P(t) + \gamma(q_0(t); t) \quad (14)$$

with the initial condition $q_1^P(0) = 0$. Correspondingly identify the complementary solution, $q_1^C(t)$, a solution of the equation

$$\dot{q}_1^C(t) = J[f(q_0(t); t)] q_1^C(t) \quad (15)$$

with the initial condition vector $q_1(t_0)$.

The solution of (15) is in the form

$$q_1^C(t) = \Phi_1(t, t_0) q_1(t_0) \quad (16)$$

where both the fundamental matrix $\Phi(t, t_0)$ and the initial condition vector $q(t_0)$ are unknown. The fundamental matrix, $\Phi(t, t_0)$ is obtained in a manner similar to that employed for the one dimensional case. The derivative of (16) is

$$\dot{q}_1^C(t) = \dot{\Phi}_1(t, t_0) q_1(t_0) \quad (17)$$

inserting (16) and (17) into (15) yields the differential equation

$$\dot{\Phi}_1(t, t_0) = J[f(q_0(t); t)] \Phi_1(t, t_0) \quad (18)$$

Equation (18) can be solved for $\Phi_1(t, t_0)$ with $\Phi_1(t_0, t_0) = I$, the identity matrix. With $\Phi_1(t, t_0)$ known we can compute the integral in (13) as the known particular solution $q_1^P(t)$. Consequently, we now

have the solution to (8) in the form

$$q_1(t) = \Phi_1(t, t_0) q_1(t_0) + q_1^P(t) \quad (19)$$

In (19) only the initial condition vector $q(t_0)$ is unknown. We solve for $q(t_0)$ by operating on the observed values of the trajectory of the solution of (8), the differential equation of the unknown system.

In expanded form we can write (19) as

(20)

$$\begin{bmatrix} q_1^1(t) \\ q_1^2(t) \\ \vdots \\ q_1^6(t) \end{bmatrix} = \begin{bmatrix} q_1^{1,P}(t) \\ q_1^{2,P}(t) \\ \vdots \\ q_1^{6,P}(t) \end{bmatrix} + \begin{bmatrix} \Phi_{1,1}^{1,1}(t, t_0) & \Phi_{1,2}^{1,2}(t, t_0) & \dots & \Phi_{1,6}^{1,6}(t, t_0) \\ \Phi_{2,1}^{2,1}(t, t_0) & \dots & \dots & \Phi_{2,6}^{2,6}(t, t_0) \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{6,1}^{6,1}(t, t_0) & \dots & \dots & \Phi_{6,6}^{6,6}(t, t_0) \end{bmatrix} \begin{bmatrix} q_1^1(t_0) \\ q_1^2(t_0) \\ \vdots \\ q_1^6(t_0) \end{bmatrix}$$

where $\{q_1^j(t)\}$ $j = 1, 2, \dots, 6$ designate set the component of the state vector $q_1(t)$, and similarly $\{q_1^{j,P}(t)\}$, $\{q_1^j(t_0)\}$ are the set of components of the particular solution and initial condition vector.

Assume that there is a single observable quantity, the displacement $y(t_i)$ of the solution of equation (2) at the different times t_i .

These observations correspond to observations of $q_1^1(t_i)$.

Our objective is to use these observations to determine the unknown initial condition vector $q(t_0)$ and hence generate a new approximation $q_1(t)$ to the solution of the original set of nonlinear differential equations.

That this can be accomplished can be seen by writing 6 equations in the first component observation $q_1^1(t_i); i = 1, 2, \dots, 6$ for the six unknown initial condition components $q_1^j(t_0), j = 1, 2, \dots, 6$ in the form

$$\begin{bmatrix} q_1^1(t_1) - q_1^{1,P}(t_1) \\ q_1^1(t_2) - q_1^{1,P}(t_2) \\ . \\ q_1^1(t_6) - q_1^{1,P}(t_6) \end{bmatrix} = \begin{bmatrix} \phi_1^{1,1}(t_1, t_0) & \phi_1^{1,2}(t_1, t_0) & \dots & \phi_1^{1,6}(t_1, t_0) \\ \phi_1^{1,1}(t_2, t_0) & \phi_1^{1,2}(t_2, t_0) & \dots & \phi_1^{1,6}(t_2, t_0) \\ . & . & . & . \\ \phi_1^{1,1}(t_6, t_0) & . & \dots & \phi_1^{1,6}(t_6, t_0) \end{bmatrix} \begin{bmatrix} q_1^1(t_0) \\ q_1^2(t_0) \\ . \\ q_1^6(t_0) \end{bmatrix} \quad (21)$$

Equation (21) can also be written as

$$[q_1^1(t_i) - q_1^{1,P}(t_i)] = \sum_{j=1}^6 \phi_1^{1,j}(t_i, t_0) q_1^j(t_0) \quad i = 1, 2, \dots, 6 \quad (22)$$

or in the vector matrix form

$$[q_1^1(t) - q_1^{1,P}(t)] = \Phi_1^{1,\cdot}(t, t_0) q_1(t_0) \quad (23)$$

Under the assumption that $\Phi_1^{1,\cdot}(t, t_0)$ is nonsingular, its inverse, $[\Phi_1^{1,\cdot}(t, t_0)]^{-1}$, exists and (23) can be solved for $q_1(t_0)$ in the form

$$q_1(t_0) = [\Phi_1^{1,\cdot}(t, t_0)]^{-1} [q_1^1(t) - q_1^{1,P}(t)] \quad (24)$$

Knowledge of $q_1(t_0)$, of (23) inserted into (19) permits the approximation $q_1(t)$ to be computed as the solution of the set of equations (8). We recall parenthetically that, $q_1(t)$ and $q_1^P(t)$, both the complementary and particular solutions to (8) are generated with the assumed solution $q_0(t)$. Consequently $q_1(t)$ is incorrect (since it is dependent upon $q_0(t)$), however, we use it to generate a new approximation $q_2(t)$ etc. Kalaba³ has demonstrated that the convergence properties of the quasilinear solution to the original nonlinear equations has quadratic convergence properties, that is, if the true solution of (8) is $q(t)$ and $q_k(t)$, $q_{k+1}(t)$ are successively the k th and $k + 1$ st approximate solutions then

$$\| q_{k+1}(t) - q(t) \| \leq M \| q_k(t) - q(t) \|^2 \quad (25)$$

where M is independent of k .

To summarize, the sequence of steps in obtaining the quasilinear solution to the nonlinear differential equations are the following:

1. Assume a solution $q_0(t)$ to the augmented linearized nonlinear differential equations
2. Solve for the fundamental matrix
3. Solve for the particular solution
4. Use the observations $y(t_i) = q^1(t_i)$ $i = 1, 2, \dots, n$ to get a 2nd approximation (estimate) of the unknown initial conditions vector $q(t_0)$.
5. Steps 2, 3, 4 give a new approximation $q_1(t)$ to the solution of the original set. Repeat steps 2, 3, and 4 until successive solutions differ to an arbitrary extent.

3.4 The Least Squares Techniques Applied To The Quasilinearization Solution.

The determination of the unknown initial condition vector and hence the identification of unknown parameters by the technique of quasilinearization suffers from the difficulty that the solution for n parameters with n observations is not necessarily unique. A simple graphical illustration of the response of a one degree of freedom linear system to a step input will illustrate the point. The response of such a system for two different degrees of damping is illustrated below.

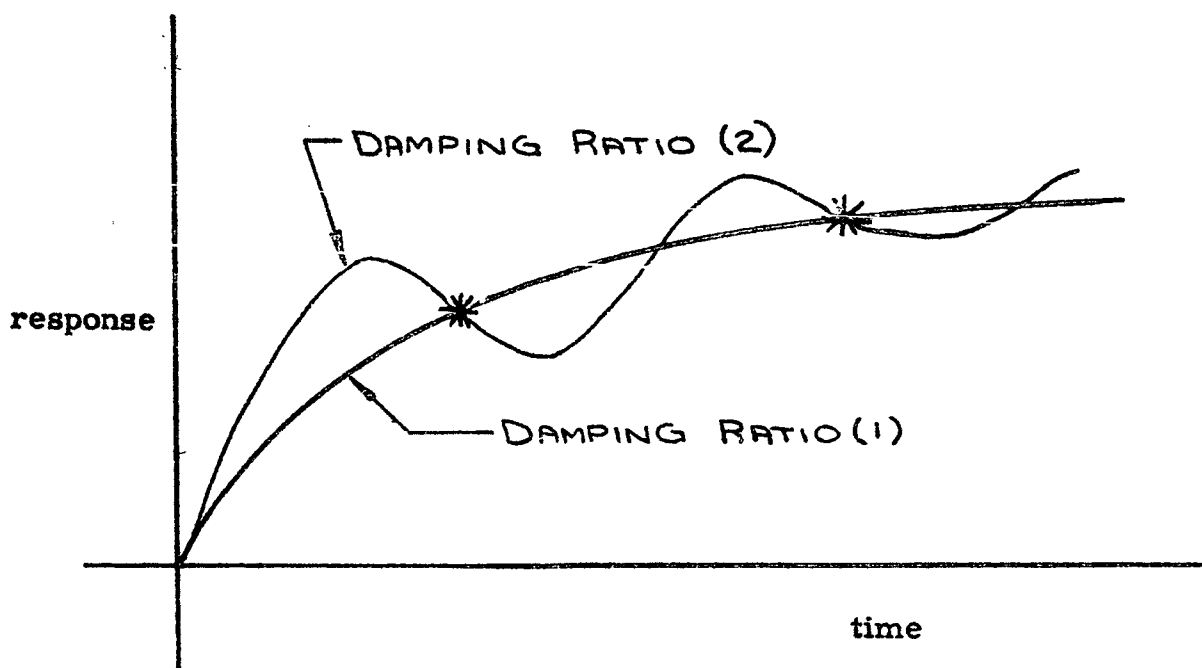


Figure 2. RESPONSE OF A ONE DEGREE OF FREEDOM LINEAR SYSTEM TO A STEP INPUT PARAMETRIC IN DAMPING RATIO.

Observe that there are a number of points of overlap between the two solutions. If, for example, the points labeled* are chosen, the technique described in Section 3.2 can be satisfied by either of the solutions drawn (as well as an infinite number of other solutions).

For this reason, as well as because of the fact that the observations of the displacement may be noisy we resort to the least squares technique of parameter estimation to permit more than the minimum number of observations (6 in the case discussed in Section 3.3) to enhance our estimate of the unknown parameters. With noisy observations of $y(t) = q^1(t)$, the model for the solution for the initial condition vector (3.3.22) becomes

$$[q_1^1(t) - q_1^{1,P}(t)] = \Phi_{1,}^{1,6}(t, t_0) q_1(t_0) + \epsilon \quad (1)$$

where ϵ is assumed to be a zero mean error vector with a covariance matrix consisting of identical diagonal elements and zeros off the diagonal (uncorrelated, equal variance observations).

In (3.3.22) the notation is used to imply that 6 observations $q_1^1(t_i)$; $i = 1, 2, \dots, 6$ are employed. The notation is quite general. In (1) above we interpret the situation as corresponding to the case in which n observations are made, in particular $n = 6$. That is, in (1) the vector $[q_1^1(t) - q_1^{1,P}(t)]$ can be interpreted as an n component observation vector, the $n \times 6$ matrix $\Phi_{1,}^{1,6}(t, t_0)$ can be interpreted a matrix of known elements, the 6×1 vector $q_1(t_0)$ is a vector of unknown parameters or components and the 6×1 vector ϵ is an error vector. The situation just described corresponds to a description of the framework for the classical least squares parameter estimation problem¹³. In the notation of that discipline equation (1)

is written

$$y = X' \beta + \epsilon \quad (2)$$

where y is a vector of observations, X' is a known transformation matrix, β is an unknown parameter vector and ϵ is as before, a zero mean vector with an equal component diagonal covariance matrix. The least square solution of (2) for the estimate β of the unknown parameter vector is

$$\hat{\beta} = [X'X]^{-1} X' y \quad (3)$$

In an equivalent form, the least squares solution to (1) (estimate of the unknown initial condition vector), with $n > 6$ observations is in the form

$$q(t_0) = [(\Phi^{1, \cdot})' \Phi^{1, \cdot}]^{-1} (\Phi^{1, \cdot})' [q^1(t) - q^{1, p}(t)] \quad (4)$$

The solution of (1) in the form (3) is seen to involve multiplication of a $n \times 6$ matrix, $(\Phi^{1, \cdot})'$ by a $6 \times n$ matrix $\Phi^{1, \cdot}$ and inversion of the resulting $n \times n$ matrix. As the amount of data gathered increases, (n increases) the cost of this computational step becomes prohibitive. In the section following we describe a stagewise smoothing procedure that eliminates the requirement that an increasingly large matrix be inverted. The procedure is referred to as stagewise estimation.

3.5 The Stagewise Estimation Procedure Applied To The Least Square Solution Of The Quasilinearization Technique

The stagewise estimation scheme, apparently originated by Swerling⁽⁷⁾ and improved upon by Ho^(8, 9) permits large amounts of data to be employed for the purpose of parameter estimation without prohibitive computational costs. In what follows the stagewise estimation procedure is demonstrated. It will be seen that other than for the first iteration, the matrix inversion step is completely eliminated. The procedure is as follows: Consider equation (3.3.22) in the form

$$y_k = X'_k \beta \quad (1)$$

where y_k is the observation vector $[q^1(t) - q^1, P(t)]$. X'_k is the transformation matrix $\Phi^{1, \cdot}(t, t_0)$ and β is the vector of unknown parameters $q(t_0)$. In (1) and all that follows the suffix k designates a k component vector or a k row matrix and also signifies that the vector and matrix are time dependent and hence index dependent.

The least squares solution of (1) is given by (3.4.3). This solution is in fact a consequence of the minimization of the quadratic form

$$Q_k = [X'_k \beta - y_k]' [X'_k \beta - y_k] = \|X'_k \beta - y_k\|^2 \quad (2)$$

with respect to β .

Consistant with the notation in (1) we write the least squares estimate (2) in the form

$$\hat{\beta}_k = [X_k X'_k]^{-1} X_k y_k \quad (3)$$

Now if additional data, represented by the m component vector, y_{k+1} , is taken we can represent the model of the observations in the partitioned matrix form

$$\begin{bmatrix} y_k \\ \vdots \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} X'_k \\ \vdots \\ X'_{k+1} \end{bmatrix} [\hat{\beta}_k + \Delta\beta_{k+1}] \quad (4)$$

In (4) $\Delta\beta_{k+1}$ represents the incremental change in the estimate of the parameter vector β due to the new data y_{k+1} . Corresponding to (2), the quadratic form to be minimized to get the new estimate of β is given by

$$Q_{k+1} = \left\| X'_k [\hat{\beta}_k + \Delta\beta_{k+1}] - y_k \right\|^2 + \left\| X'_{k+1} [\hat{\beta}_k + \Delta\beta_{k+1}] - y_{k+1} \right\|^2 \quad (5)$$

Using the fact that $X'_k \hat{\beta}_k = y_k$, the first term on the right hand side of (5) can be written as $\Delta\beta'_{k+1} X'_k X'_k \Delta\beta_{k+1}$. The second term can be systematically expanded. Equation (5) is to be minimized with respect to $\Delta\beta_{k+1}$ the incremental change in the estimate of β . This equation consists of quadratic terms in the form $z'Az$ and linear terms in the form Bz . Differentiation of these terms with respect to the vector gives the partial results

$$\frac{\partial}{\partial z} z'Az = 2Az$$

$$\frac{\partial}{\partial z} Bz = B$$

Application of these formulas to the expanded version of (5) and forming the equation

$$\frac{\partial Q_{k+1}}{\partial \Delta \beta_{k+1}} = 0 \quad (6)$$

yields the result

$$\Delta \hat{\beta}_{k+1} = [X_k X_k' + X_{k+1} X_{k+1}']^{-1} X_{k+1} [y_{k+1} - X_{k+1}' \beta_k] \quad (7)$$

Equation (7) is an updating scheme for improving the estimate $\hat{\beta}_k$ as more observations y_{k+1} are made. Now we employ a result used by Ho to eliminate the successive inversion of matrices as indicated by (7). Let

$$[X_k X_k']^{-1} = A_k \quad (8)$$

$$[X_k X_k' + X_{k+1} X_{k+1}']^{-1} = A_{k+1}$$

Then

$$A_{k+1}^{-1} = A_k^{-1} + X_{k+1} X_{k+1}' \quad (9)$$

Equations (9), and (8) applied to (7) yield the stagewise estimation procedure

$$\Delta\beta_{k+1} = A_{k+1} X_{k+1} (y_{k+1} - X'_{k+1} \beta_k) \quad (10)$$

where

$$A_{k+1} = A_k - A_k X_{k+1} (X'_{k+1} A_k X_{k+1} - I)^{-1} X'_{k+1} A_k \quad (11)$$

Equation (11) is seen to involve an inversion of an $m \times m$ matrix where m is arbitrary. In particular we make $m = 1$ and eliminate the inversion process.

4. ILLUSTRATIVE COMPUTATIONAL RESULTS AND EXPECTED EFFORT IN THE SECOND QUARTER

Preliminary computations have been made on the identification of a simulated one degree of freedom linear system. Independent samples of a zero mean gaussian random variable are used as a forcing function into a one degree of freedom linear system represented by a differential equation. The response of the system to the forcing function is computed using a Runge-Kutta numerical integration procedure.

The assumed unknown mass, spring and damping parameters of the system are estimated by the quasilinearization--least squares--stagewise smoothing procedure. The example illustrated consisted of an underdamped system that satisfies the equation

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad (1)$$

Equation (1) is written in the form

$$\ddot{x} + \alpha_1 \dot{x} + \alpha_2 x = \alpha_3 \ddot{f}(t) \quad (2)$$

We identify the following state variables

$$\begin{aligned} y_1 &= x \\ y_2 &= \dot{x} \\ y_3 &= \alpha_1 \\ y_4 &= \alpha_2 \\ y_5 &= \alpha_3 \end{aligned} \quad (3)$$

which leads to the following set of differential equations

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -y_4 y_1 - y_3 y_2 + y_5 \ddot{f}(t) \\ \dot{y}_3 &= 0 \\ \dot{y}_4 &= 0 \\ \dot{y}_5 &= 0 \end{aligned} \quad (4)$$

In vector-matrix form (4) can be expressed as the equation

$$\dot{y} = g(y, t) \quad (5)$$

The $k + 1$ at iterative equation corresponding to (5) is

$$\dot{y}_{k+1} = \left. \frac{\partial g}{\partial y} \right|_{y=y_k} y_{k+1} + \{ g(y_k, t) - \left. \frac{\partial g}{\partial y} \right|_{y=y_k} y_k \} \quad (6)$$

The original true parameter values chosen were

$$\begin{aligned} \alpha_1 &= 5 \\ \alpha_2 &= 25 \\ \alpha_3 &= 1 \\ x(0) &= y_1(0) = 0 \\ \dot{x}(0) &= y_2(0) = 0 \end{aligned} \quad (7)$$

The original guesses were

$$\begin{aligned} x_0(t) &= y_{10}(t) = \sin \sqrt{10}t \\ \dot{x}_0(t) &= y_{20}(t) = \sqrt{10} \cos \sqrt{10}t \\ \alpha_1 &= 10 \\ \alpha_2 &= 10 \\ \alpha_3 &= 10 \end{aligned} \quad (8)$$

The response of the simulated system is computed at .050 second intervals, the observations used for quasilinearization computation (5 are required) are taken at .5 second intervals, observations used for the stagewise smoothing are also taken at .5 second intervals.

Five observations are taken.

A table of the evolution of the estimate of the unknown parameters is provided below.

TABLE 1. EVOLUTION OF THE PARAMETER ESTIMATES

| Parameter | True Value | 1st Quasi-linearization | Stagewise* Smoothing (1) | Stagewise** Smoothing (2) | Stagewise*** Smoothing (4) |
|------------|------------|-------------------------|--------------------------|---------------------------|----------------------------|
| $x(0)$ | 0 | 0.000 | 0.000 | 0.000 | 0.000 |
| $x(0)$ | 0 | 0.031 | -0.012 | -0.017 | -0.000 |
| α_1 | 5 | 9.959 | 9.970 | 6.333 | 4.979 |
| α_2 | 25 | 10.231 | 10.114 | 23.851 | 25.000 |
| α_3 | 1 | 0.943 | 0.919 | 1.018 | 1.004 |

where *, **, *** designates the column of estimates obtained after a total of 10 observations have been on respectively the 1st, 2nd and 4th iteration. The procedure gave true estimates, to within 3 decimal points after the 7th iteration.

Figure 1 is a graph of the response of the simulated system compared with the response of the system computed using the parameter estimates given by the first quasilinearization and stagewise smoothing iteration. These graphs correspond to computations for the true parameters and * in Table 1. Similarly, Figure 2 corresponds to computations for the true parameters and for ** in that Table.

The results presented are preliminary. They represent a first attempt at verifying the computational procedure. The procedure appears to work satisfactorily. Additional parameter runs to determine sensitivity of the computations to the original guess and runs with multidegree of freedom systems are contemplated for the next quarterly interval.

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0.132E 01

$$AX1 = 0.9890E-02 \quad AX2 = 0.5613E-01 \quad AX3 = 0.$$

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- * = true response of simulated system
- = response by quasilinearization



2

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-0.480E 00 0.120E 00 0.720E 00 0.132E 01 0.1

PERFORMANCE INDEX QUASI = $0.1378E-05$

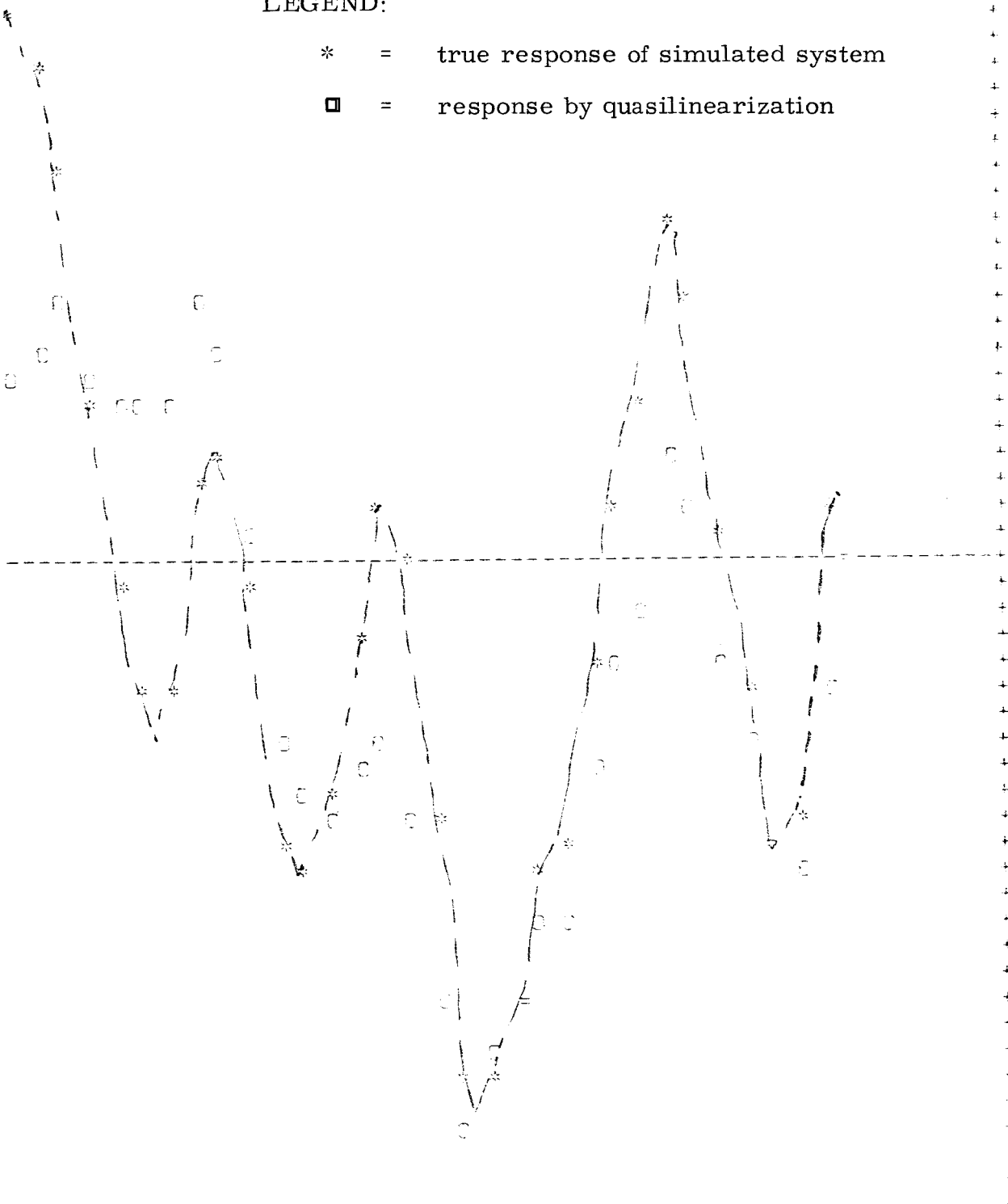
NUMBER OF POINTS USED FOR ESTIMATE = 5

AX1 = $0.4868E-02$ AX2 = $0.3120E-01$ AX3 = 0.

AX4

.....

- * = true response of simulated system
- = response by quasilinearization



| | | | | |
|--------|--------|--------|--------|--------|
| 0.1025 | 0.2525 | 0.3125 | 0.4325 | 0.5525 |
| 0.1025 | 0.2525 | 0.3125 | 0.4325 | 0.5525 |